

Growth and crisis, unavoidable connection?

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Abstract

In emerging economies periods of rapid growth and large capital inflows can be followed by sudden stops and financial crises. In the presence of financial markets imperfections, a simple modification of a neoclassical growth model can account for these facts. I study a growth model for a small open economy where decreasing marginal returns to capital appear after the country has reached a threshold level of development, which is uncertain. Limited enforceability of contracts allows default on international debt. International investors optimally choose to suddenly restrict lending when decreasing marginal returns appear. The economy defaults and enters a financial crisis.

Keywords: growth, sudden stop, default, financial crisis.

JEL Classification Numbers: F32, F34, O41, O47.

1 Introduction

This paper shows that, in the presence of financial market imperfections, sudden stops and financial crises can simply be the natural outcome of a typical growth process with decreasing marginal returns to capital.

Extensions of the neoclassical growth model to the case of a small open economy (Barro, Mankiw, Sala-I-Martin [1995], Lucas [1990]) predict that, as long as the marginal return to capital is higher than the international interest rate, a developing country borrows from the rest of the world to finance investment. As the economy grows, decreasing marginal returns eventually drive to zero the spread between the international interest rate and the return on the country's capital. The GDP reaches a steady state, while positive current

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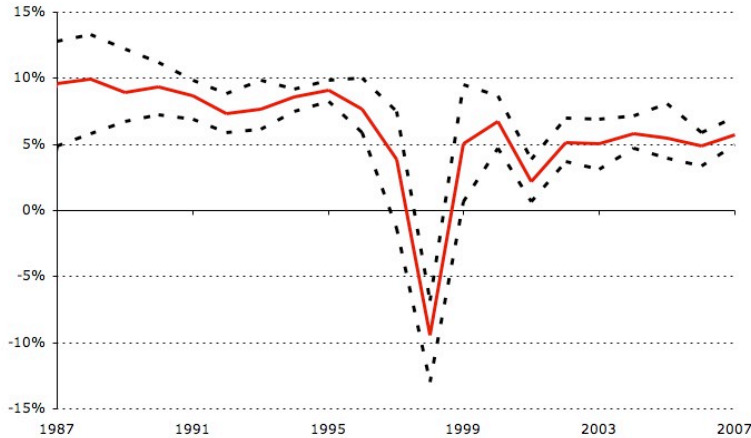


Figure 1: GDP Growth Rate for South Korea, Thailand, Indonesia and Malaysia. Average (solid), maximum and minimum (dashed).

accounts ensure that the debt accumulated with the rest of the world is repaid. In reality, financial markets imperfections can seriously impair the borrowing ability of a country, and expose it to financial crises. For instance, the theoretical analysis of Bulow and Rogoff (1989) emphasizes that limited enforceability of debt contracts can have the extreme consequence of completely preventing international lending. On the empirical side, financial openness and capital flows to developing countries are accompanied by sudden stops and financial crises (Calvo [1998]), especially for countries with very high growth rates (Ranciere, Tornell, Westermann [2008]).

A striking example of rapid GDP growth followed by a sudden stop is the case of the Asian crisis in the late '90s, which is among the deepest financial crises of the past 20 years. Figure 1 depicts the GDP growth rate of four Asian countries in the years 1987-2007. The year of the crisis is clearly marked by a fall in the GDP of about 10% in most of the economies. Capital inflows, that had financed current account deficits and fueled high growth in the first half of the decade, stopped (Figure 2). The post-crisis years see a permanent reduction in the growth rate, from an yearly 8% to about 5%, and a collapse in the investment rate (Figure 3). These patterns are confirmed in wider samples of countries that have experienced financial crises (Lee and Rhee[2000] and Ranciere, Tornell, Westermann [2008]).

Sudden stops are a recurrent features of emerging countries. Much of the existing literature rationalizes sudden stops as consequences of business cycle shocks (Mendoza [2001]), herding (Cole and Kehoe [2000]) and exogenous shocks to trend productivity growth (Aguiar and Gopinath [2007], Aguiar and Gopinath [2004]). The goal of this paper is to provide an alternative and complementary framework, which emphasizes how financial crises can be the natural outcome of a process of long run growth with decreasing marginal returns, provided that two conditions are met. The first condition is that there is uncertainty on the country's production technology, which creates uncertainty on the timing with which decreasing marginal returns appear. The second is that international financial markets are imperfect due to limited enforceability of debt contracts.

In particular, I construct an analytically tractable model of growth, where the growth path

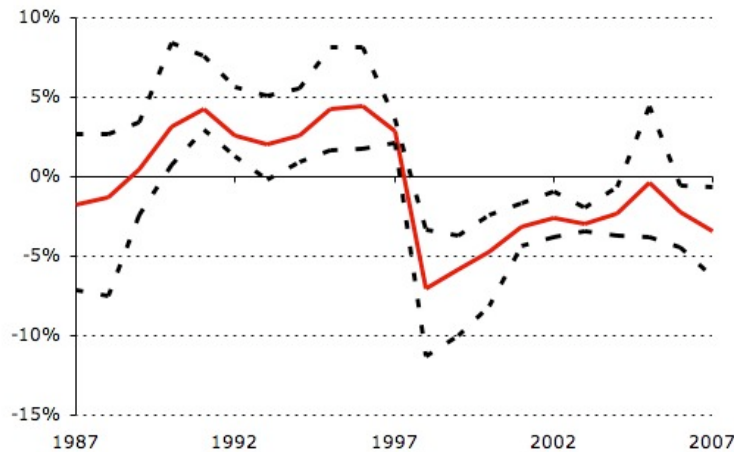


Figure 2: Capital account in percentage of GDP for South Korea, Thailand, Indonesia and Malaysia. Average (solid), maximum and minimum (dashed).

of a country is divided into two successive stages: in the first, marginal returns to capital are constant, while in the second they decrease over time. Decreasing marginal returns appear when the economy reaches an uncertain threshold level of development, called “turning point”. Households that live in the country have access to international financial markets and at any time have the option to default on their debt obligations. The punishment for default is stochastic autarky during which households suffer output costs. Because of the default possibility, international investors impose borrowing constraints to the economy. These constraints are endogenously obtained in equilibrium, and represent the highest amount of debt that, at any given state, the household can commit to repay. Endogenous borrowing constraints have been introduced in the literature in various contexts (Kehoe and Levine [1993], Alvarez and Jermann [2000], Chatterjee, Corbae, Rios-Rull [2007]). Here I provide a novel and analytically simple treatment of endogenous borrowing constraints extending standard growth models such as Barro, Mankiw, Sala-I-Martin (1995). Under appropriate parametrizations, I prove that the borrowing constraint during the first stage is large and binding.

The country experiences strong capital inflows, which fuel high growth and high investment rates (credit and growth boom). However, when the turning point is reached, international lenders restrict the supply of credit generating a sudden stop, which then triggers a financial crisis (bust). After the crisis, borrowing constraints are permanently tighter, and this permanently reduces growth and investment rates, and improves the current account. The arrival of the turning point clearly brings “bad news” about the country’s technology. I show that, in order to trigger the crisis, it is enough that the value of these “bad news” is arbitrarily small, in the sense the news could well refer to an arbitrarily distant future. In this sense, the technological shock may be interpreted as “news shock” (Cochrane [1994]). From a policy point of view, the paper investigates whether the imposition of controls on capital inflows is beneficial. I show that, for an interesting set of parameters where a simple proof is possible, capital controls avoid the boom-bust cycle, but reduce welfare.

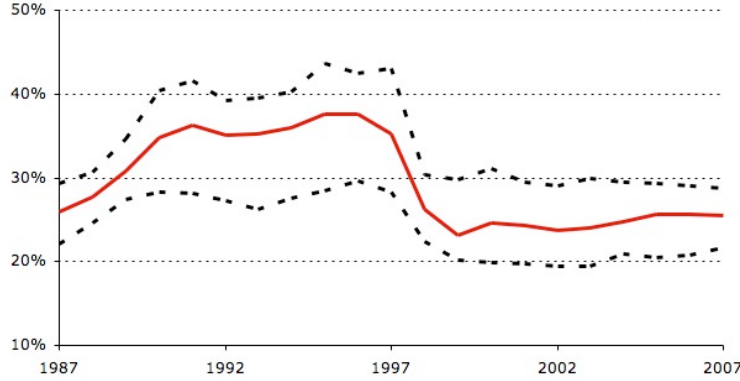


Figure 3: Investment in percentage of GDP for South Korea, Thailand, Indonesia and Malaysia. Average (solid), maximum and minimum (dashed).

Finally, the equilibrium growth rate of the economy in the first stage can be large enough to exceed the interest rate paid on the international debt. The evolution of the country's debt then follows bubble-like dynamics and non-autarkic equilibria are possible even when, as in Bulow and Rogoff (1989), output costs for defaulters are absent.

The remaining of the paper is organized as follows. Section 2 introduces the technologies, the problem of the household and of the international investors. Section 3 defines an equilibrium. Section 4 defines and characterizes an equilibrium with endogenous borrowing constraints. Section 5 summarizes the main results of the paper. Section 6 shows the connection between bubbles, growth and debt sustainability. Section 7 discusses some model's extensions. Section 8 concludes.

2 The model

2.1 Technology

In this section I introduce the properties of the production technology and I intuitively explain how it generates a growth process that can be divided into two stages. In the first, the economy grows along a balanced path with constant marginal returns to capital. This stage ends, and the second begins, when the economy reaches a threshold level of development called *turning point*. During the second stage, the economy keeps growing but the marginal return on capital progressively falls over time. The threshold turning point is given by the realization of a random variable.

Time is continuous and the instantaneous production of output Y_t takes place according to a constant return to scale technology

$$Y_t = K_t^\alpha H_t^{1-\alpha} \quad (1)$$

Aggregate capital K is accumulated at the household level in the usual linear fashion $\dot{K}_t = X_t - \delta K_t$, where X_t is aggregate investment and $\delta > 0$ is the depreciation rate. The input H ,

instead, cannot be accumulated and is supplied to the production sector at a price $w(H; \bar{H})$,

$$w(H; \bar{H}) = \begin{cases} \underline{w} & H \leq \bar{H} \\ \underline{w} + \gamma \left[\left(\frac{H}{\bar{H}} \right)^{1+\gamma} - 1 \right] & H > \bar{H} \end{cases} \quad (2)$$

for strictly positive constants γ , \underline{w} and \bar{H} . The supply of H is perfectly elastic up to the level \bar{H} , the turning point. The assumption of infinite elasticity is applied, for instance, by Lewis (1954) to model the industrial revolution: a large pool of farmers live in the countryside where they gain a subsistence wage \underline{w} , which is also the reservation value at which they are willing to move to the cities and supply labor to the industrial sector. Differently from Lewis (1954), however, I assume that the pool of these “subsistence farmers” is of a finite size \bar{H} . Consequently, once \bar{H} has been reached, the wage rate must rise if more workers are to be attracted to the industrial sector. This example is part of a more general phenomenon that takes place when economic growth is driven by increased utilization of production inputs, such as labor, human capital (Chari and Hopenhayn [1991]), land and other natural resources, that are initially not fully employed. As growth unfolds, more of these idle inputs progressively enter the production process without generating any factor price increase. At a certain point, the pool of idle inputs is exhausted and further expansions in their utilization requires increases in their price. The turning point can also be thought of as a moment where the economy hits a technological barrier (as in Parente and Prescott [1994]).

There is a fundamental form of uncertainty in the economy, which surrounds the value of \bar{H} . More specifically, at time zero agents know that \bar{H} is drawn from a Pareto distribution with cumulative density function given by,

$$\text{Prob}(\bar{H} \leq H) = 1 - \left(\frac{H_0}{H} \right)^\eta \quad (3)$$

where $\eta > 0^1$ and $H_0 > 0$ is the aggregate amount of factor H used for production at time zero. Uncertainty on \bar{H} implies that the moment at which H_t reaches the turning point \bar{H} is also unknown.

Assumptions (1) and (2) deliver the classical idea (Marshall [1898]) that, without an offsetting technical progress, economic growth is eventually accompanied by a fall in the marginal return to capital. To make this point, recall that when markets are competitive the marginal return to the factor H must equal its price w_t ,

$$w_t = (1 - \alpha) \left(\frac{K_t}{H_t} \right)^\alpha$$

The marginal return to capital r_t^k is then,

$$r^k(w_t) = \alpha \left(\frac{K}{H} \right)^{\alpha-1} = \alpha \left[\frac{1 - \alpha}{w_t} \right]^{\frac{1-\alpha}{\alpha}} \quad (4)$$

¹For $\eta \in (0, 1)$, the Pareto distribution has no finite first and second moment. This is of no consequence in the model. Low values for η give a distribution with a fat tail, which create beliefs that place a lot of mass on large realizations of \bar{H} .

During the first stage of growth, where $w_t = \underline{w}$, the marginal return to capital is constant and then $G_{H,t} \equiv \frac{\dot{H}_t}{H_t} = \frac{\dot{K}_t}{K_t} \equiv G_{K,t}$. Since \bar{H} is stochastic, the time \tilde{t} at which H_t reaches the turning point is also uncertain and is given implicitly by

$$\bar{H} = H_0 e^{\int_0^{\tilde{t}} G_{H,t} dt} \quad (5)$$

Now suppose that at some time t the turning point has not been reached ($H_t < \bar{H}$). Conditional on this knowledge, the probability that the turning point is reached within time $t + \epsilon$ is obtained from Bayes' rule as

$$\begin{aligned} \text{Prob}\{\bar{H} \leq H_t e^{\int_t^{t+\epsilon} G_{H,\tau} d\tau} | H_t < \bar{H}\} &= \frac{\text{Prob}\{H_t < \bar{H} < H_t e^{\int_t^{t+\epsilon} G_{H,\tau} d\tau}\}}{\text{Prob}\{H_t < \bar{H}\}} \\ &= 1 - e^{-\eta \int_t^{t+\epsilon} G_{H,\tau} d\tau} = \eta G_{H,t} \epsilon + o(\epsilon) \end{aligned}$$

As ϵ goes to zero we see how assumption (3) implies that the timing of the turning point is exponentially distributed with arrival rate π_t proportional to $G_{H,t}$, that is

$$\pi_t = \eta G_{H,t} \quad (6)$$

In the paper we will be dealing with the situation where the economy follows a balanced growth path before the turning point, with all aggregate variables growing at a common rate G^* . Equation (6) would then guarantee that the arrival rate of the turning point is constant $\pi = \eta G^*$.

I make the following convenient assumptions on the marginal return to capital in the first stage of growth,

$$0 < r^k(\underline{w}) - \delta - \rho < \rho + \theta \quad (7)$$

for some parameters $\rho > 0$ and $\theta \geq 0$ defined in the next section. The lower bound on $r^k(\underline{w}) - \delta - \rho$ guarantees that the economy has a positive growth process. The upper limit, instead, is chosen for analytical tractability since it sets a convenient bound to the growth rate of the economy.

2.2 Households

There is a unit measure of infinitely lived households, who derive instantaneous utility $u(c_t) = \ln c_t$ from consumptions c_t . Households carry on production at the individual level according to the constant returns to scale technology (1), using their individual capital stock k_t and buying inputs h_t . Moreover, they invest x_t and make additions d_t to their outstanding stock of debt b_t , financed by international investors (discussed in the next section). Households decide, at any point in time, whether or not to default on their debt. The choice of defaulting entails the double punishment of exclusion from borrowing opportunities and a cost which reduce the household's output by a fraction ξ_t . The presence of output costs as part of the default punishment (as in Arellano [2008]) have been justified by Mendoza and Yue (2008). The default punishment ends stochastically in an i.i.d. fashion across defaulting households. The individual stock of outstanding debt b_t of a household in the non default

state is constrained by $b_t \leq m_t k_t$, where m_t is a leverage limit exogenous to the household's choice.

To introduce the household's dynamic optimization problem, we need first to define an appropriate state space, given by an individual state s_t and an aggregate state S_t .

To keep the exposition at a tractable level, I employ a parsimonious version of the aggregate state, given simply by the price w_t , i.e. $S_t = w_t$. A vector $\mathbf{P}(S) = (\pi(S), G_S(S))$ of aggregate policies determines the Markov process for the aggregate state, in the following way. The state S_t is constant ($S_t = \underline{w}$) from time zero to the turning point time \tilde{t} . The turning point time \tilde{t} is exponentially with state dependent arrival rate $\pi(S_t)$,

$$\pi(S_t) = \begin{cases} \pi & S_t = \underline{w} \\ 0 & S_t > \underline{w} \end{cases} \quad (8)$$

With a slight abuse of notation, we indicate with a constant² $\pi > 0$ the arrival rate of the turning points, while $\pi(S) = 0$ indicates that at states $S > \underline{w}$ the turning point has already occurred. After the turning point, the state S_t grows at a certain rate $G_S(S_t)$. To guarantee that in equilibrium $G_S(S_t) > 0$, so that the economy goes through the turning point only once, it will be enough to assume that, at every $\xi(S)$, the output cost function $\xi(S)$ satisfies³

$$[1 - \xi(S)]r^k(S) - \delta - \rho \geq 0 \quad (9)$$

where $r^k(S)$ is the marginal product of capital defined in (4). Moreover, for simplicity, I require that the fraction of output lost in autarky does not increase over time, i.e. $\xi'(S) \leq 0$. Borrowing constraints $m(S)$ depend on the aggregate state. I progressively refine the set of functions $m(S)$ that we are interested in considering. In this section, I only require that $m(S)$ has at most a finite number of discontinuities. In Section 2.3 I show that international investors' rationality places further restrictions on $m(S)$; borrowing constraints satisfying those conditions are called *sustainable*. The final refinement, presented in Section 4, leads us to the definition of *endogenous borrowing constraints*. Discontinuities on the borrowing constraint function are an important element in our analysis. In particular, define the function $m_+(S)$ as the limit from the right of $m(S)$,

$$m_+(S) \equiv \lim_{\epsilon \downarrow 0} m(S + \epsilon) \quad (10)$$

I will say that there is a *credit crunch* at state S if $m_+(S) < m(S)$, meaning that the borrowing constraint is suddenly tightened. A case of particular interest is when $m_+(\underline{w}) < m(\underline{w})$, so that there is a sudden stop at the turning point.

The household's individual state is summarized by the vector $s_t = (k_t, b_t, j_t)$, where $j_t \in$

²Combining (6) and (8) we see that focusing on Markov equilibria restricts us to equilibria with a balanced growth rate $G = \frac{\rho}{\pi}$ before the turning point. This choice, while greatly simplifying the exposition, is not very restrictive since, as explained in Section 4, only forces us to choose appropriate initial conditions to set the economy at time zero on its balanced growth path.

³As we will see, this is indeed equivalent to requiring that the equilibrium growth rate of aggregate capital is always non negative. The condition will also be interpreted as assuming that output costs, arising from financial autarky, cannot be so high as to reduce the household's net return on capital below the international risk free rate.

$\{D, ND\}$ is an indicator specifying, respectively, whether the household has reached time t is in the default state or not. The household's choice to instantaneously repay or default on the debt is represented by a policy $j'(s, S)$, with $j' \in \{D, ND\}$.

Consider an individual state $s_{ND} = (k, b, ND)$. Let $V^{j'}(s_{ND}, S)$ indicate the value to a household initially in the non default state who takes the instantaneous decision to default ($j' = D$) or not to default ($j' = ND$). The optimal choice $j'^*(s_{ND}, S)$ gives, at any state, the optimal value to the household $V(s_{ND}, S) = \max\{V^{ND}(s_{ND}, S), V^D(s_{ND}, S)\}$. If the household is indifferent between $j' = D$ and $j' = ND$ I break the tie assuming $j'^* = ND$. Moreover, if $j'^*(s_{ND}, S) = D$ I say that the individual state j switches from ND to D at (s_{ND}, S) .

At any state state s_D such that $s_D = (k, b, D)$ the autarky punishment forces the household to a choice $j'^*(s_d, S) = D$ and thus $V(s_D, S) = V^D(s_D, S)$. The default state ends stochastically with an arrival rate θ . When this happens j switches from D to ND and the household's previous debt is erased, implying that default is complete⁴. The interest rate $r(s, S)$ paid by the household on her outstanding debt incorporates a default risk and presented in Section 2.3.

While the aggregate state is a continuous process, the individual stocks k and b can have a finite number of discontinuities, called "jumps". The stocks k and b change continuously at (s, S) if and only if their corresponding controls $x(s, S)$ and $d(s, S)$ are finite valued. At such states the Bellman equations for $V^{ND}(s_t, S_t)$ and $V^D(s_t, S_t)$ satisfy⁵

$$[\rho + \pi(S)]V^{ND}(s, S) = \max_{x,d,h} \log c + \pi(S)V_{TP}(s, S) + V_S \dot{S} + V_k \dot{k} + V_b \dot{b} \quad (12)$$

$$\text{s.t. } \begin{aligned} c &= Ak^\alpha h^{1-\alpha} - w(S)h + d - x \\ \dot{k} &= -\delta k + x \\ \dot{b} &= r(s, S)b + d \\ b &\leq m(S)k \end{aligned}$$

⁴In Section 7.2 I present the case of partial default.

⁵In what follows it is understood that $V_k^{j'} = \partial V^{j'} / \partial k$, $V_b^{j'} = \partial V^{j'} / \partial b$ and $V_S^{j'} = \partial V^{j'} / \partial S$. If $w_t = \underline{w}$, then the economy is in the first stage of growth and $\frac{\partial V^{j'}}{\partial w} \dot{w} = 0$. Since the debt stock doesn't change in the default state we have $V_b^D \dot{b} = 0$.

Moreover, in the definitions of the value functions it is implicitly assumed that if default is not optimal at some time t then it is not optimal at $t + \varepsilon$ (provided that the turning point does not occur between t and $t + \varepsilon$). In other words, we are assuming that,

$$V_t = V_t^{ND} \Rightarrow V_{t+\varepsilon} = V_{t+\varepsilon}^{ND} \quad \tilde{t} \notin [t, t + \varepsilon] \quad (11)$$

The condition above would be clearly true if the values V_t^{ND} and V_t^D were continuous functions of t except, possibly, at the turning point \tilde{t} . The simplest way to justify restriction (11) is that (11) turns out to be a condition that must hold for an equilibrium to exist. In fact, condition (11) is necessary for the borrowing constraints to be "sustainable", as shown in Section 2.3. Restricting our-selves to equilibria where (11) holds is then, for our purposes, without loss of generality.

and

$$[\rho + \theta + \pi(S)]V^D(s, S) = \max_{x^D, h} \log c^D + \theta V^{ND}(k, 0, ND, S) + \pi(S)V_{TP}(s, S) + V_S^D \dot{S} + V_k^D \dot{k} \quad (13)$$

$$\text{s.t. } c^D = [1 - \xi(S)][Ak^\alpha h^{1-\alpha} - w(S)h] - x^D \\ \dot{k} = -\delta k + x^D$$

The function V_{TP} gives the value at the turning point. The turning point brings “bad news” to the household, since it indicates that from that moment on the price w will rise, and it is then associated with a sudden downwards revision in the household’s value⁶. More precisely, $V(s, S)$ is in general discontinuous at $S = \underline{w}$. The correct way to define V_{TP} is then by setting

$$V_{TP}(s, \underline{w}) = \lim_{\epsilon \downarrow 0} V(s, \underline{w} + \epsilon) \quad (14)$$

When the investment and debt flows $x^*(s, S)$ and $d^*(s, S)$ in (12) have no finite solution, then we say that there is a jump in k and b at (s, S) . An intuitive example of how we deal with these jumps is the following. Suppose that at some state (s_{ND}, S) we have $b < m(S)k$. Since the borrowing constraint is not binding, the household might choose to adjust instantaneously her debt to a higher level $b_+ = b + \Delta$ and, correspondingly⁷, finance an increase in her capital stock to $k_+ = k + \Delta$, where $\Delta > 0$. Clearly, feasibility requires that the jump Δ must still satisfy the borrowing constraint $b_+ \leq m(S)k_+$ and thus

$$\Delta \leq \frac{m(S)k - b}{1 - m(S)} \quad (15)$$

Negative jumps $\Delta < 0$ are also possible, as we shall see in the next section, and represent situations in which the household sells part of her capital to instantaneously repay some of her outstanding debt.

Define $\Delta^*(s, S)$ the optimal jump policy for k and b . At states where $\Delta^*(s, S) = 0$ the individual variables k and b move continuously and their evolution is given, respectively, by $x^*(s, S)$ and $d^*(s, S)$ in (12)-(13). The individual policy function is summarized by the vector $\mathbf{p}^*(s, S)$,

$$\mathbf{p}^*(s, S) = (x^*(s, S), d^*(s, S), j'^*(s, S), \Delta^*(s, S), h^*(s, S))$$

⁶As I explain in Section 5, the main results of the paper don’t hinge on the “size” of these “bad news”, i.e. on the size of the downward revision at the turning point of the household’s value V .

⁷Appendix A shows how the model with jumps is formally derived by solving (12) and (13) with an additional constraint \bar{D} on the maximum instantaneous growth of debt, and then taking the limit for $\bar{D} \rightarrow \infty$. Moreover, it is shown that if jumps Δ^b in debt and Δ^k in capital are optimal at some state then $\Delta^b = \Delta^k = \Delta$.

2.3 International Investors

Households borrow from competitive international investors, who are risk neutral and discount time at rate ρ . Investors take as given the evolution of the aggregate state and the household's optimal policies. In particular, investors are aware of the household's default policy $j^*(s, S)$. In this section I show that, in order for the borrowing constraints $m(S)$ to be meaningfully defined, default can be optimal only at the turning point. Using this result, I also provide simple expressions for the interest rate function $r(s, S)$.

To develop intuition, think now of a discretized version of the model with time intervals $\epsilon > 0$. Assume that the household defaults when, at a certain time $t > \tilde{t}$ after the turning point, the state (s_{ND}, S) is reached. Recall that after the turning point all the uncertainty in the economy is resolved and then the household's default at t would be a deterministic event from the perspective of some time $t - \epsilon$. Any rational investor would avoid lending to the household at time $t - \epsilon$, i.e. the "moment before" default occurs. Hence, borrowing constraints can represent actual levels of debt that investors are willing to supply only if default is never optimal after the turning point,

$$j^*(k, b, ND, w) = ND \quad \forall b \leq m(S)k, S > \underline{w} \quad (16)$$

Now consider the case where the household defaults at some time $t < \tilde{t}$ before the turning point, when the state (s_{ND}, \underline{w}) is reached. At time $t - \epsilon$ investors know that (s_{ND}, \underline{w}) is to be reached approximately with probability $1 - \pi\epsilon$. A different state (s, S) with $S > \underline{w}$ is instead reached if, with probability $\pi\epsilon$, the turning point is realized in the time span ϵ . As ϵ goes to zero, the probability of reaching the default state (s_{ND}, \underline{w}) goes to one. Once again, default is a deterministic event, and investors would not lend to the household at time $t - \epsilon$. The borrowing constraint $m(\underline{w})$ can represent actual levels of debt that international investors are willing to supply only if

$$j^*(k, b, ND, \underline{w}) = ND \quad \forall b \leq m(\underline{w})k \quad (17)$$

Borrowing constraints that satisfy (16) and (17) are called *sustainable*, and satisfy a minimum rationality property of the supply of credit by international investors. We have shown that when borrowing constraints are sustainable, default cannot happen at a time t strictly after ($t > \tilde{t}$) or strictly before ($t < \tilde{t}$) the turning point \tilde{t} . Can default happen exactly at \tilde{t} ? The answer is affirmative since, as noted above, at time $\tilde{t} - \epsilon$ investors would assign only a vanishing probability $\pi\epsilon$ to the default event. In particular, consider the credit crunch in Figure 4. If, at the arrival of the turning point, the household's accumulated capital and debt are such that $b \leq m_+(\underline{w})k$ then it cannot be optimal to default, since equation (16) gives, in the limit, $j^*_+(k, b, ND, \underline{w}) = ND$, where

$$j^*_+(s, \underline{w}) \equiv \lim_{\epsilon \downarrow 0} j^*(s, \underline{w} + \epsilon)$$

But, in Figure 4, $m(\underline{w}) > m_+(\underline{w})$ so it is also possible that the accumulated k and b at the turning point are such that $b > m_+(\underline{w})k$. At the turning point, the household could choose to default ($j^* = D$). Alternatively, in order not to default ($j^* = ND$) but at the same

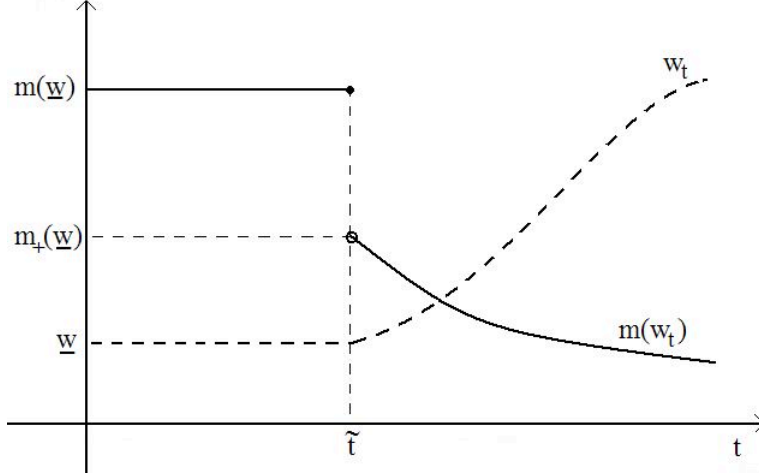


Figure 4: Example of time evolution of borrowing constraints featuring a credit crunch at the turning point \tilde{t} .

time to satisfy the new (tighter) borrowing constraint, she would have to instantaneously sell an amount Δ of her capital to repay part of her debt and make it jump down to $b + \Delta \leq m_+(\underline{w})(k + \Delta)$, for some $\Delta < 0$.

It is now easy to provide an explicit form for the interest rate function $r(s, S)$. The zero profit condition for international investors implies that

$$r(s_{ND}, S) = \begin{cases} \rho & S > \underline{w} \\ \rho & S = \underline{w} \text{ and } j_+^*(s_{ND}, S) = ND \\ \rho + \pi & S = \underline{w} \text{ and } j_+^*(s_{ND}, S) = D \end{cases} \quad (18)$$

A default premium equal to the arrival rate π is charged to debt levels on which default is optimal at the turning point. Lending policies are given by the vector $\mathbf{I}(s, S) = (m(S), r(s, S))$.

3 Equilibrium

Any policy $\pi(\underline{w})$ is associated with an unique constant growth rate $G(\underline{w})$ in (6), implying that any equilibrium must be characterized by a balanced growth path in the first stage of growth. Given an initial condition H_0 and a balance growth G , then (5) uniquely pins down the timing $\tilde{t}^{\bar{H}}$ of the turning point for a given realization of \bar{H} . After $\tilde{t}^{\bar{H}}$ the evolution of the aggregate state is uniquely determined by the policy $G_S(S) > 0$. In conclusion, we can associate to any \bar{H} a unique deterministic path $\{S_t^{\bar{H}}\}_0^\infty$ for the aggregate state. Aggregate and individual policies must be consistent in equilibrium. In particular, given initial condition $s_0(i) = (k_0(i), b_0(i), ND)$ for each household $i \in [0, 1]$, the individual optimal policies $\mathbf{p}(s, S)$ determine a unique equilibrium process⁸ for every individual state $\{s_t^{\bar{H}*}(i)\}_0^\infty$ for all i . Aggregating across households, the individual policy function $h^*(s, S)$ determines

⁸The process may not be deterministic, since households that default along the equilibrium path face idiosyncratic punishment durations. Idiosyncratic shocks are irrelevant for aggregation purposes.

a deterministic path $\{H_t^{\bar{H}^*}\}_0^\infty$ for the aggregate factor H as follows

$$H_t^{\bar{H}^*} = \int_{[0,1]} h^* \left(s_t^{\bar{H}^*}(i), S_t^{\bar{H}} \right) di \quad (19)$$

Definition 1 *An equilibrium is given by an aggregate policy vector $\mathbf{P}(S)$, a lending policy vector $\mathbf{l}(s, S)$, an individual policy vector $\mathbf{p}(s, S)$ and individual initial conditions $\{s_0(i)\}_{i \in [0,1]}$ such that*

- i) Given $\mathbf{P}(S)$ and $\mathbf{l}(s, S)$, the policies $\mathbf{p}(s, S)$ solve the household's problem.*
- ii) Given $\mathbf{P}(S)$ and $\mathbf{p}(s, S)$, the borrowing constraints in $\mathbf{l}(s, S)$ are sustainable and the interest rate schedule satisfies (18).*
- iii) Given initial conditions, aggregate and individual policies are consistent, i.e.*

$$S_t^{\bar{H}} = w \left(H_t^{\bar{H}^*}; \bar{H} \right) \quad \forall \bar{H}, t$$

where the function $w(\cdot; \cdot)$ is given by (2), $H_t^{\bar{H}^*}$ is given by (19), and $\{S_t^{\bar{H}}\}$ is the process induced by the aggregate policies.

The household's problem can be significantly simplified. First, by equating w to the marginal product of h we have that

$$h^*(s, w) = \left[\frac{A(1-\alpha)}{w} \right]^{\frac{1}{\alpha}} k \quad (20)$$

Equation (20) tells us that the correct initial belief H_0 in (3) satisfies

$$H_0 = \left[\frac{A(1-\alpha)}{\underline{w}} \right]^{\frac{1}{\alpha}} \int_{[0,1]} k_0(i) di$$

Substituting $h^*(s, S)$ in the household's production function and defining the marginal product of capital $r^k(S)$ as in (4) we rewrite the budget constraints in (12) and (13), respectively, as

$$\begin{aligned} c &= r^k(S)k + d - x \\ c^D &= [1 - \xi(S)]r^k(S)k - x^d \end{aligned}$$

The linearity of the constraints in k and the homotheticity of the utility functions allows us to redefine the household's problems (12)-(13) in a detrended form. For any state s , define $\hat{s}(s) = (\hat{b}, j)$ the detrended individual state, where $\hat{b} = \frac{b}{k}$ is the household's leverage. Call $g = \frac{\dot{k}}{k} = \frac{x}{k} - \delta$ the individual capital growth rate. It is easy to guess and verify that⁹

$$V^{j'}(s, S) = \hat{V}^{j'}(\hat{s}, S) + \frac{1}{\rho} \ln k \quad (21)$$

⁹Appendix A provides details on the derivation of (21). Notice that once (21) is established we have $j'^*(s, S) = j'^*(1, \hat{s}, S)$ so that (22) is verified from (18).

$$r(s, S) = r(1, \hat{s}, S) \quad (22)$$

where the detrended value $\hat{V}^{ND}(\hat{s}, S)$ solves

$$\begin{aligned} [\rho + \pi(S)]\hat{V}^{ND}(\hat{s}, S) &= \max_{g, \hat{d}} \log \hat{c} + \pi(S)\hat{V}_{TP}(\hat{s}, S) + \hat{V}_b^{ND}\dot{\hat{b}} + \hat{V}_S^{ND}\dot{S} + \frac{g}{\rho} \\ \text{s.t. } \hat{c} &= r^k(S) - \delta - g + \hat{d} \\ \dot{\hat{b}} &= [r(1, \hat{s}, S) - g]\hat{b} + \hat{d} \\ \hat{b} &\leq m(S) \end{aligned}$$

and $\hat{V}^D(\hat{s}, S)$ solves

$$\begin{aligned} [\rho + \theta + \pi(S)]\hat{V}^D(\hat{s}, S) &= \max_{g^D, \hat{d}} \log \hat{c}^D + \theta\hat{V}(1, 0, ND, S) \\ &\quad + \pi(S)\hat{V}_{TP}(\hat{s}, S) + \hat{V}_S^D\dot{S} + \frac{g^D}{\rho} \\ \text{s.t. } \hat{c}^D &= \xi(S)r^k(S) - \delta - g^D \end{aligned}$$

The detrended values \hat{V} , \hat{V}_{TP} and the individual policy vector $\hat{\mathbf{p}}$ are correspondingly defined over the new state space (\hat{s}, S) .

As an instructive example, take the autarkic borrowing constraints $m(S) = 0$ and initial conditions $b_0(i) = 0$. These borrowing constraints are obviously sustainable, and thus $j^*(s_{ND}, S) = ND$ at all states. In any equilibrium, the evolution of r^k after the turning point is linked to the instantaneous growth rate of the aggregate capital $G_K = \frac{\dot{K}}{K}$ by the following differential equation

$$\frac{\dot{r}^k}{r^k} = -a(r^k)G_K \quad (23)$$

where $a(r^k) > 0$ is a continuous function¹⁰. The first order conditions in the detrended household problem give $g^* = r^k - \delta - \rho$ and $c^* = \rho$. The equilibrium path of the economy is then completely characterized by a balanced growth path $G^* = g^* = r^k(\underline{w}) - \delta - \rho$ in the first stage of growth, a random turning point $\tilde{t}^H = (G^*)^{-1}(\log \bar{H} - \log H_0)$, an evolution of r^k in the second stage governed by the first order differential equation (23) in the variable r^k , with $G_K^*(r^k) = r^k - \delta - \rho$ and initial condition $r^k(\underline{w})$. Marginal returns to capital asymptotically decrease to $\rho + \delta$ and capital accumulation, and thus economic growth, slows down towards zero.

4 Equilibrium with endogenous borrowing constraints

The autarkic equilibrium, presented in the previous section, shows that the simple requirement that borrowing constraints be sustainable might lead to the selection of equilibria where borrowing constraints are somewhat “too tight”. To make this point, assume that borrowing constraints are autarkic and take some states S and $s_{ND} = (k, b, ND)$ with $b > 0$,

¹⁰For the derivation of (23) and of $a(r^k)$ see Appendix C.

$k > 0$ and $\xi(S) > 0$. Intuitively, if b is sufficiently close to zero, then the household's utility cost of not defaulting, and instantaneously adjusting her debt down to the autarkic constraint $m = 0$ (by choosing $\Delta^* = -b$), is smaller than the utility cost of defaulting. Formally, $j^*(s_{ND}, S) = ND$ and the household would not default at (s_{ND}, S) . The autarkic borrowing constraints are then too tight in the sense that they prevent the household from choosing debt levels $b > 0$ that could nonetheless be sustained in equilibrium. This situation seems unnatural, since it prevents international investors and households to enter financing contracts that could be beneficial for both parties.

To obviate this problem, we focus on borrowing constraints $m^*(S)$, defined endogenously, that have the property of being “not too tight”,

Definition 2 *An equilibrium is said to have endogenous borrowing constraints $m^*(S)$ if at all S the following property holds*

$$\hat{j}^*(\hat{b}, ND, S) = D \quad \forall \hat{b} > m^*(S) \quad (24)$$

Borrowing constraints $m^*(S)$ are *endogenous* in the sense that they are equilibrium objects, since they depend on the equilibrium optimal policies (24). Endogenous borrowing constraints are “not too tight” in the sense that they exclude only debt contracts that would have never been signed anyway: no international investor would ever agree to leverage a household beyond $m^*(S)$ since the household would, with probability one, immediately default.

5 Results

The prototypical growth model for a small open economy with exogenous borrowing constraints, as in Barro, Mankiw and Sala-I-Martin (1995), predicts a “smooth” development path for a country. This is in stark contrast with the evidence in Figure 1-2. Our model with endogenous borrowing constraints helps to bridge this gap between theory and evidence. Specifically, the model predicts that, depending on the parameters value, a small open economy will follow either a “boom-bust” growth process, marked by a sudden stop and a financial crisis, or a standard “smooth” process.

The characterization of endogenous borrowing constraints entails solving a fixed-point problem. Specifically, for any given borrowing constraints function $m(\cdot)$, the household's production choices determine the time evolution of the aggregate state S_t . In turn, the time evolution of S_t influences the household's optimal default policy $\hat{j}^*(s, S)$, thus determining whether the constraints $m(\cdot)$ initially conjectured can indeed be labeled endogenous. As I show in the rest of the section, the solution to this fixed-point problem has a simple analytical characterization once we consider the following property,

Proposition 1 *Borrowing constraints $m^*(S)$ are endogenous if and only if, in equilibrium,*

$$\hat{V}^{ND}(m^*(S), ND, S) = \hat{V}^D(m^*(S), ND, S) \quad (25)$$

Proof. See Appendix B. ■ Endogenous borrowing constraints are fully characterized by the property of making households indifferent between defaulting and not defaulting. Proposition

1 thus generalizes, to the case of a production economy, the results in Eaton and Gersovitz (1981) or Hellwig and Lorenzoni (2009).

Whether the economy follows a boom-bust or a smooth growth path depends on the model's parameters,

Proposition 2 *Given appropriate initial conditions, an equilibrium with endogenous borrowing constraints always exists. There is always a credit crunch at the turning point, $m_+^*(\underline{w}) < m^*(\underline{w})$, and $m^*(S)$ is continuous and decreasing at states $S > \underline{w}$. Moreover, there exist threshold values $\underline{\eta} < \bar{\eta} < 1$ such that*

- 1) for $\eta \leq \underline{\eta}$, $m^*(S)$ always optimally binds. All households default at the turning point (boom-bust equilibrium).
- 2) for $\eta \geq \bar{\eta}$, $m^*(S)$ optimally binds only at $S > \underline{w}$, while at $S = \underline{w}$ households choose the constant leverage $\hat{b}^* = m_+^*(\underline{w})$. No household ever defaults (smooth equilibrium).
- 3) $\underline{\eta}$ is independent on the parameter γ in equation (2).

Proof. See Appendix E. ■ An appropriate initial condition is needed for the existence of an equilibrium, and boils down to making sure that the economy at time zero is already set on its initial balanced growth path¹¹. As a parallel to Figure 1-3, Figure 5 reports the evolution for the GDP growth¹² rate G_Y^* , the capital account to GDP ratio $\frac{D^*}{Y^*}$, and the investment to GDP ratio $\frac{X^*}{Y^*}$ for the two types of equilibria. In a smooth equilibrium the variables follow the standard path as in Barro, Mankiw and Sala-I-Martin (1995). The situation, instead, is quite more dramatic in a boom-bust equilibrium. The initially greater leverage $b^*(\underline{w}) = m^*(\underline{w}) > m_+^*(\underline{w})$ boosts capital inflows, investment rates and growth to higher levels than those of a smooth equilibrium. Households initially choose to leverage all the way up to $m^*(\underline{w})$, but then they face a sudden tightening of the borrowing constraint at the turning point. Rather than sell their capital to satisfy the new and lower constraint $m_+^*(\underline{w})$, households choose to default. Lack of access to the financial market and output costs force output, investment and the capital account to drop. As households idiosyncratically regain access to the financial market, all the variables improve, but they settle down to much lower levels than those before the turning point. Part of the reason is that returns to capital are now marginally lower. But more important is the fact that the credit crunch is permanent (Figure 4) and, correspondingly, the lower leverage permanently reduces growth, investment

¹¹Our parsimonious choice of the aggregate state $S = w$ excludes the possibility of transitional dynamics to the initial balanced growth occurring up to the turning point. Therefore, to guarantee the existence of an equilibrium, we need to select an initial condition that “sets” the economy immediately on its balanced growth path. This is achieved by choosing appropriate initial distributions of capital $\{k_0(i)\}_{i \in [0,1]}$ and debt $\{b_0(i)\}_{i \in [0,1]}$ across households that match the equilibrium leverage $\hat{b}^*(\underline{w})$ on the balanced growth path. In a boom-bust equilibrium the correct choice is then $b_0(i) = m^*(\underline{w})k_0(i)$, and in a smooth equilibrium $b_0(i) = m_+^*(\underline{w})k_0(i)$.

¹²In a boom-bust equilibrium, the GDP at the turning point time \tilde{t} instantaneously drops from $Y_{\tilde{t}}$ to $Y_{\tilde{t}+} = [1 - \xi(\underline{w})]Y_{\tilde{t}}$, and thus the growth rate $G_{Y,t}^* = \frac{\dot{Y}_t}{Y_t}$ is not defined at \tilde{t} . In particular, $\lim_{t \downarrow \tilde{t}} \frac{Y_t - Y_{\tilde{t}}}{t - \tilde{t}} = -\infty$. To address this graphical issue, stemming from the continuous-time nature of our model, and allow a direct comparison with Figure 1, the GDP growth rate in Figure 5 is computed over discretized intervals of arbitrary length $\epsilon > 0$, so that $G_{Y,t}^* = \frac{Y_{t+\epsilon} - Y_t}{Y_t}$ in Figure 5.

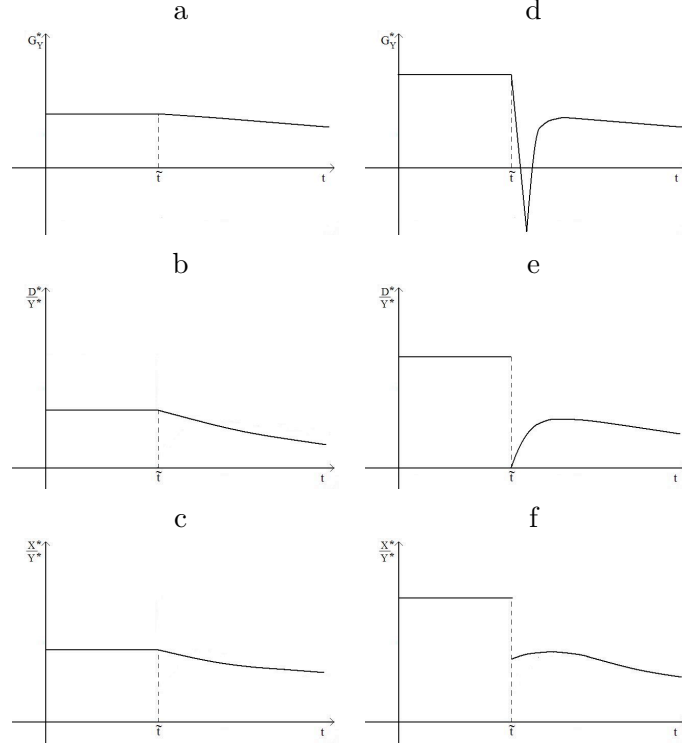


Figure 5: GDP growth G_Y^* , capital account-GDP ratio $\frac{D^*}{Y^*}$ and investment rate $\frac{X^*}{Y^*}$ in a smooth equilibrium (a-c) and in a boom-bust equilibrium (d-f).

rates and the capital account.

To gain intuition for Proposition 2, it is useful to treat separately its two main results. The first is that endogenous borrowing constraints always feature a credit crunch at the turning point, and behave exactly as in Figure (4). The second is that, when η is small, the economy follows a boom-bust equilibrium with $m^*(\underline{w})$ binding, while for η large the economy is on a smooth equilibrium with $m^*(\underline{w})$ not binding.

Let us start with the first result, which is better understood when we focus on boom-bust equilibria, where the effects of the credit crunch fully play out. In equilibrium, the spread between the net marginal return to capital and the interest rate paid by households on their debt is always positive,

$$(r^k - \delta) - r^* > 0 \quad (26)$$

Inequality (26) is an intuitive arbitrage condition that has to hold in order to provide the households with incentives to borrow from abroad and invest at home. Moreover, the indifference condition (25) for a household with leverage $m^*(S)$ is equivalent to

$$\log \hat{c}^* + \frac{g^*}{\rho} = \log \hat{c}^{*D} + \frac{g^{*D}}{\rho} + \frac{\theta}{\rho} \log \frac{1}{1 - m^*} \quad (27)$$

where optimal capital growth rates g^* , g^{*D} and (detrended) consumptions \hat{c}^* , \hat{c}^{*D} respectively in the non-default and default states are obtained at any instant from the household's first

order conditions,

$$g^* = \frac{(r^k - \delta) - r^* + \dot{m}^*}{1 - m^*} + r^* - \rho \quad (28)$$

$$g^{*D} = (1 - \xi)r^k - \delta - \rho \quad (29)$$

$$\hat{c}^* = \rho(1 - m^*) \quad (30)$$

$$\hat{c}^{*D} = \rho \quad (31)$$

Details on the derivation of (26) and (27)-(31) are given in Appendix E. The left-hand side of (27) is the flow of value to a household that chooses to stay in the contract and repay her debt, and is equal to the flow of utility from consumption plus the increase g^* in the household's assets (capital), discounted by the rate of time preference ρ . The right-hand side is the flow of value to a household that defaults, which equals the utility from consumption and the growth in assets in the default state, plus a third term. The term captures the discounted value of the instantaneous growth $\frac{1}{1-m^*}$ in the household's net worth, stemming from the fact that, at the rate θ , the household is readmitted to the financial market with its prior debt erased (net worth then goes up from $(1 - m^*)k$ to k). Using (30)-(31) it is easy to verify that (27) can hold only if $g^* > g^{*D}$; the non-default state is characterized by greater capital accumulation, and this is why it features a lower detrended consumption $\hat{c}^* < \hat{c}^{*D}$. Equation (27) says that a large endogenous leverage m^* can be sustained only if a high growth g^* counterbalances the loss of utility from a reduced consumption \hat{c}^* and from a lower net worth. By (28), high growth requires a large large spread $(r^k - \delta) - r^*$ (notice, also, that before the turning point leverage is constant and then $\dot{m}^* = 0$). Moreover, crucially, growth and leverage are *self-reinforcing*: while a large growth is needed to sustain a large leverage m^* , a large leverage in turns magnifies the positive effect on growth of a large spread $(r^k - \delta) - r^*$. Intuitively, when leverage is large, the household can achieve a high investment rate. Since her capital stock is expanding fast, her borrowing constraint m^*k is also expanding fast. In this situation, debt is to a great extent rolled-over at every instant, reducing the household's incentives to default. The self-reinforcing property of growth and leverage is a powerful mechanism, which gives rise to equilibria where the country's debt grows fast enough to form a bubble (see Section 6).

Before the turning point, r^k is high making it possible to sustain high growth and a large endogenous leverage. However, as the economy surpasses the turning point, the ability of the self-reinforcing property to sustain a high leverage breaks down. As marginal returns r^k decrease, growth g^* slows down, reducing households' incentives not to default. Equation (27) then requires that, in parallel to the reduction in r^k , leverage m^* must also be reduced, namely $\dot{m}^* < 0$ (recall also that $r^* = \rho$ after the turning point). Since the de-leveraging process further chokes-off growth, it cannot be too pronounced, otherwise (27) would stop holding and the household would choose to default. For instance, from (28) an obvious bound on \dot{m}^* is $\dot{m}^* > -(r^k - \delta - \rho)$ at any time. Endogenous borrowing constraints after the turning point must then make sure both that leverage m^* is low when marginal returns r^k eventually become low, and both that the de-leveraging process is not too sharp. This double requirement can be met only if $m_+^*(\underline{w})$, the borrowing constraint at the beginning of the de-leveraging process, is not too large.

In conclusion, before the turning point, a high and steady marginal return r^k allows a large growth g^* , which positively self-reinforces the ability of the economy to sustain an initial large and constant $m^*(\underline{w})$. After the turning point, instead, falling marginal returns progressively slows down growth g^* , negatively self-reinforcing the dynamics of m^* . The necessary de-leveraging process requires an initial sudden downwards adjustment of the borrowing constraint to a lower level $m_+^*(\underline{w}) < m^*(\underline{w})$, creating a credit crunch.

The second aspect of Proposition 2 that we need to explain is why the size of the parameter η determines whether the economy follows a boom-bust or a smooth equilibrium. A household that, before the turning point, chooses a leverage $\hat{b}^*(\underline{w})$ such that $m_+^*(\underline{w}) < \hat{b}^*(\underline{w}) \leq m^*(\underline{w})$ is charged a default premium $\pi^* = \eta G^*$ on her debt. Instead, if $\hat{b}^*(\underline{w}) \leq m_+^*(\underline{w})$, the household pays only the risk-free rate $r^* = \rho$. The equilibrium spread $(r^k - \delta) - r^*$ is strictly positive, and thus it is always optimal for the household to be as much leveraged as possible for a given r^* . The household's problem is then simply to decide between the two options $b^*(\underline{w}) = m_+^*(\underline{w})$ or $b^*(\underline{w}) = m^*(\underline{w})$ (incidentally, this also explains why $m^*(S)$ always binds after the turning point). Households have to choose whether to limit their arbitrage activity to a lower leverage m_+^* but take advantage of a larger spread $(r^k - \delta) - \rho$, or to increase their leverage to $m^* > m_+^*$ but face a smaller spread $(r^k - \delta) - \rho - \pi^*$. The former choice is optimal if η , and thus π^* , are high, while the latter is optimal if η is small.

When η is smaller, the tail of the distribution for \bar{H} is fatter, and the country has a better “fundamental” for long-term growth. Our discussion then implies that countries with better fundamentals for long-term growth are more likely to face high initial growth and capital inflows (with possible bubble-like characteristics, as in Section 6), interrupted by a sudden stop and a financial crisis. On the contrary, countries with worse fundamentals, attract less capital, grow less rapidly, and do not go through a financial crisis¹³.

When the turning point is reached, “bad news” are revealed to the economy. Larger values for γ are associated with “worse news”, since marginal returns fall more quickly. Instead, in the limit for $\gamma \rightarrow 0$, bad news become arbitrarily small and the economy spends, after the turning point, an arbitrarily long time with marginal returns still close to $r^k(\underline{w})$. Interestingly, point 3) shows that the size of γ (the “size” of the “bad news”) does not matter to determine whether a boom-bust occurs. Therefore, in our model, it is not the size of a technological shock that determines the extent of a sudden stop, as it is for instance in Aguiar and Gopinath (2007) or Arellano (2008). This result obtains because even slowly decreasing marginal returns force households on a de-leveraging path. Low values of $\gamma > 0$ only buy households more time before they are required to attain a significant de-leveraging. But when this moment is reached, the (negatively) self-reinforcing effect of growth and leverage would kick-in, and households would default. Once the turning point is reached, this future default time would be perfectly foreseen by international investors, who then immediately restrict their credit supply, generating the sudden stop and triggering the default event¹⁴.

¹³If $\underline{\eta} < \eta < \bar{\eta}$ the equilibrium is a “mix” between the two pure cases of boom-bust and smooth. For $\underline{\eta} < \eta < \bar{\eta}$ households are indifferent in equilibrium between choosing the higher leverage $m^*(\underline{w})$ or the lower leverage $m_+^*(\underline{w})$. A certain endogenous fraction ϕ^* chooses the higher leverage and defaults at the turning point, and the remaining fraction $1 - \phi^*$ chooses the lower leverage and doesn't default. Appendix E explains how ϕ^* is computed.

¹⁴Steps 1-2 in Appendix E provide an analytical and graphical explanation for why $\gamma > 0$ does not play any role in determining the occurrence of a boom-bust equilibrium.

Is it optimal, from a welfare perspective, to allow an economy to follow a boom-bust path, or would it be better to force it on a smooth equilibrium? A smooth equilibrium can always be artificially achieved if the government imposes capital controls that limit to $m_+^*(\underline{w})$ the maximum leverage of households before the turning point. The answer to the above question is difficult to provide in general, but there is a case of particular interest where we can show that capital controls turn out to be sub-optimal, and households are better off in a boom-bust equilibrium.

Consider an economy which, in the absence of capital controls, would follow a boom-bust equilibrium and denote with $m^{*BB}(S)$ the usual endogenous borrowing constraints. Call $m^{CC}(S)$ the borrowing constraints that have the property of being endogenous after the turning point, i.e. $m^{CC}(S) = m^{*CC}(S)$ for $S > \underline{w}$, but limit leverage to $m^{CC}(\underline{w}) = m_+^{*CC}(\underline{w})$ before the turning point. If $\theta = 0$ we can prove the following,

Proposition 3 *Call $\hat{V}_{BB}(\hat{b}, ND, \underline{w})$ and $\hat{V}_{CC}(\hat{b}, ND, \underline{w})$ the (detrended) optimal household's values, respectively, in the boom-busts equilibrium and in the capital control equilibrium. Then, for $\theta = 0$,*

$$\hat{V}_{BB}(\hat{b}, ND, \underline{w}) > \hat{V}_{CC}(\hat{b}, ND, \underline{w})$$

for any $\hat{b} \leq m^{CC}(\underline{w})$.

Proof. See Appendix D. ■ When $\theta = 0$, default entails permanent exclusion from the financial market and permanent output costs. This is the case where the punishment from default is the toughest, and hence this is the situation where one might expect the capital control policy to be most effective in improving welfare. Contrary to this intuition, Proposition 3 shows that capital controls that avoid the crisis reduce welfare. The reason is that the benefits of a fast development process in the first stage, made possible by the large leverage and capital inflows, outweigh the costs of having to go through a sudden stop and the associated crisis.

6 Debt and bubbles

In their influential paper, Bulow and Rogoff (1989) showed that no contingent debt contract can be enforced between a lender and a borrower whenever the only punishment for default is the permanent exclusion from future borrowing. The model that I have constructed gives an instructive example of how the Bulow-Rogoff (from now BR) result is strictly connected to a transversality condition that rules out bubbles.

The formulation of BR is easily replicated here by setting $\theta = 0$ and $\xi(S) = 0$ for all S . We can show that, consistently with the BR result¹⁵,

$$m^*(S) = 0 \quad \forall S > \underline{w}$$

However, as stated in Proposition 2, we find can $m^*(\underline{w}) > 0$ if η is sufficiently small. Why does the BR result seem to fail here? In this case, after substituting for the first order

¹⁵An easy way to see this is by noting that $m_t = 0$ solves (38) in Appendix E for $\xi_t = 0$.

conditions, equation (27) is equivalent to

$$\rho[\log(1 - m^*) + m^*] - m^*(\rho + \pi^* - G^*) = 0$$

where as usual $G^* = \pi^*/\eta$. Clearly, we can have a solution $m^* > 0$ only if

$$\rho + \pi^* - G^* < 0 \tag{32}$$

The balanced growth path prior to default has strictly positive debt only if the interest rate $r^* = \rho + \pi^*$ is smaller than the the balance rate G^* . The household can then continuously roll over her current debt obligations by issuing new debt.

Following BR, define $D^0(T)$ as the time zero *present value of the net payment* flows d_t^* up to time T from the international investors to the small open economy. After the turning point \tilde{t} the economy is in autarky, then $d_t = 0$. Rewriting the law of motion of the household's debt in terms of leverage we obtain that, before the tuning point,

$$0 = (\rho + \pi^* - G^*)m^*(\underline{w}) + \hat{d}_t^*$$

where, as usual, $\hat{d}_t = \frac{d_t}{k_t}$ is the detrended debt flow. The present value $D^0(T)$ is then

$$D^0(T) = E \left[- \int_0^T e^{-\rho t} \hat{d}_t^* k_t^* dt \right] = (\rho + \pi^* - G^*)m^*(\underline{w})k_0 \int_0^T e^{-(\rho + \pi^* - G^*)t} dt$$

where the expectation for the time \tilde{t} after which the flows $\hat{d}_t = 0$ are zero is computed using the arrival rate π^* . Notice that $D^0(\infty) = -\infty$, meaning that the usual transversality conditions does not hold here. These equilibria are ruled out by BR, who restrict themselves to considering only cases where the present value of the payments is non negative. Equilibria with rational bubbles can be obtained in a number of ways (see for instance Weil [1989], Jovanovic [2007], Hellwig and Lorenzoni [2009]). Interestingly, for the existence of an equilibrium with rational bubbles it is generally necessary that the bubble asset has at any time a bounded supply (Jovanovic [2007]). In our model, the bound on the amount of debt that the country can issue is endogenously obtained because incentives to default limit at any time to $m^*(\underline{w})$ the household's leverage.

7 Discussion and extensions

7.1 Comparison with other models of sudden stops

Various models have tried to rationalize the presence of sudden stops and financial crises in emerging economies. These models rely mainly either on business cycle shocks (as in Arellano [2008]), or on trend productivity shocks (Aguilar and Gopinath. [2007]). The model presented in this paper shares, with the aforementioned literature, common observational predictions about the path of growth rates, productivity and capital accounts. However, it suggests very different causal relations between these variable.

Business cycle shocks are usually measured by Solow residuals (TFP). In the model, the

TFP calculated across the households' production technologies at any time t is

$$TFP_t = (1 - \phi_t^*)A + \phi_t(1 - \xi_t)A = (1 - \xi_t\phi_t^*)A$$

where ϕ_t^* is the endogenous fraction of aggregate capital owned by households in the default state¹⁶. The time series of the TFP in a boom-bust equilibrium would show a constant value A up to the crises, then a sharp fall down to $[1 - \xi(\underline{w})]A$ at the turning point ($\phi_t^* = 1$) and a subsequent smooth reversion to the long run value A ($\phi_t^* = 0$ asymptotically). In the model, the default event is then associated with a fall in productivity, but is not caused by it (as in Arellano [2008]). In fact, the causality runs in the opposite direction, because it is the sudden stop that forces the households to default and thus to bear the output costs (Mendoza [2008] makes a similar point).

Causalities are also reversed when we compare the model with Aguiar and Gopinath. (2007). In the latter, a shock to output growth, due to a trend shock to productivity, generates a sudden stop. Here, instead, it is the sudden stop, with the permanent reduction in leverage, which causes a trend shock to output growth.

Finally, as noted in Section 5, in our model the sudden stop is not generated by a large shock to the current aggregate technology. In fact, in the model, endogenous borrowing constraints sharply adjust to news revealed at the turning point even when these news refer to technological conditions (lower marginal returns) that take place arbitrarily far in the future.

7.2 Extension

In this section I consider three generalizations of the model: non-Pareto distributions for the threshold \bar{H} , decreasing marginal returns also before the turning point, and partial default.

Consider any probability distribution function $F(\cdot)$ for \bar{H} with infinite support. A first order Taylor expansion gives

$$\text{Prob} \left\{ H_t < \bar{H} < H_t e^{\int_t^{t+\epsilon} G_{H,t} d\tau} \right\} = \hat{\eta}(H_t) G_{H,t} \epsilon + o(\epsilon)$$

where $\hat{\eta}(H) = \frac{F'(H)}{1-F(H)} H$. In a model with a general probability distribution for \bar{H} we would have then to include H as a state variable to derive the arrival rate policy $\pi(S) = \hat{\eta}(S) G_H(S)$. To obtain a boom-bust equilibrium it is crucial that, at least asymptotically, $\hat{\eta}(H)$ is small enough. Good candidates for this property are fat tail distributions¹⁷, for which we have

$$\lim_{H \rightarrow \infty} \hat{\eta}(H) = \eta$$

¹⁶The equilibrium law of motion for ϕ_t is given in Appendix E

¹⁷Fat tail distributions, used in extreme value theory, are defined as having the property that, for some constant c and $\eta > 0$,

$$1 - F(H) = cH^{-\eta} + o(H^{-\eta})$$

for some constant $\eta > 0$. The Pareto distribution gives, in fact, the tail behavior of fat tail distributions.

The assumption that w , and thus the marginal return to capital, is constant in the first stage of growth can also be dispensed of. All we need is that $w(H; \bar{H})$ is convex in H and satisfies,

$$w(H; \bar{H}) \leq \underline{w} \quad \forall H, \bar{H} > 0, H < \bar{H}$$

As long as the economy has not reached the turning point ($H < \bar{H}$), the price w must be bounded above by some constant \underline{w} , for any possible realization of \bar{H} . This preserves the uncertainty, before the turning point, about the time when marginal returns to capital eventually fall below $r^k(\underline{w})$. The presence of this uncertainty, which makes default a random event, is vital for the the existence of a boom-bust equilibrium. As an example, take

$$w(H; \bar{H}) = \underline{w} \frac{H}{1+H} \cdot \max \left\{ 1, \frac{H}{\bar{H}} \right\}$$

for $\underline{w} > 0$. Here, marginal return to capital are strictly decreasing even during the first state of growth. The endogenous borrowing constraint would not be constant anymore during the first stage where, but would still have to satisfy (27).

The extension to a debt re-negotiation environment with partial default is straightforward. The question to answer is the following: at the turning point, what is the maximum debt b_{max} that international investors can re-negotiate with an household under the threat of the usual default punishment? By the very definition of endogenous borrowing constraints, the straightforward answer is $b_{max} = m_+^*(\underline{w})k$, where k is the household's capital at the turning point. In a boom-bust equilibrium, the households would now partially default only on the fraction $\zeta^* = \frac{m^*(\underline{w}) - m_+^*(\underline{w})}{m^*(\underline{w})}$ of her debt. The appropriate default premium would then now be $\pi\zeta^*$.

8 Conclusions

I have constructed a model that links, along the growth path of a small open economy, the growth rate of output, the capital flows and the possibility of a financial crisis. Households of the small economy borrow from international investors to finance the capital accumulation which drives economic development. Default on the international debt is possible, determining the presence of endogenous borrowing constraints. Marginal returns to capital are initially constant but start to decrease when the economy reaches an exogenously given level of development that I call "turning point". The interaction of the borrowing constraints with the growth process gives rise to interesting dynamics whenever we introduce a simple element of uncertainty, i.e. agents have only a probabilistic knowledge of the level of the turning point. Two paths of development are possible, depending on the model's parametrization. In the boom-bust case, borrowing constraints are initially large and binding, and the economy grows very rapidly. When the turning point is reached, the economy is hit by a sudden stop and a financial crisis. The country eventually recovers from the distressed periods, but the greater tightness of the borrowing constraints permanently reduces the growth of output and the investment rate. In a smooth equilibrium, instead,

the endogenous borrowing constraints are not initially binding. This produces a slower but more stable growth, and no crisis takes place at the turning point. The occurrence of a crisis at the turning point does not depend on the “size” of the technological shock at the turning point, namely it does not depend on how quickly marginal returns fall. Policies that impose constraints on capital inflows avoid financial crises but can reduce welfare. I show that there is a strict connection between debt sustainability and the existence of equilibria featuring bubbles. Overall, the paper provides an extension of standard neoclassical growth models to the case of a small open economy facing endogenous borrowing constraints. In this context, a novel explanation for the occurrence of financial crises is presented. Such crises are triggered simply by financial market imperfections and a growth characterized by decreasing marginal returns.

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Appendix A

This appendix is divided in two parts. In the first I show how to obtain the husehold's detrended problem. In the second I given an example that shows how the model with jumps is derived as a limit of a model with continuous policies. For concision, I provide these derivations only at states $S > \underline{w}$ (the case $S = \underline{w}$ is very similar).

Deriving the detrended model. Notice that,

$$\dot{\hat{b}} = \frac{\dot{b}}{k} - \hat{b}g$$

Guess that (21) holds. We have

$$V_k^{ND}(k, b, ND, S)\dot{k} = -\hat{V}_{\hat{b}}(\hat{b}, ND, S)\hat{b}g + \frac{g}{\rho}$$

$$V_b^{ND}(k, b, ND, S)\dot{b} = \hat{V}_{\hat{b}}(\hat{b}, ND, S)\frac{\dot{b}}{k}$$

From which,

$$V_k^{ND}\dot{k} + V_b^{ND}\dot{b} + V_S^{ND}\dot{S} = \hat{V}_{\hat{b}}(\hat{b}, ND, S)\dot{\hat{b}} + \hat{V}_S(\hat{b}, ND, S) + \frac{g}{\rho}$$

Substituting into the definition of V^{ND} we obtain

$$\rho V^{ND}(k, b, ND, S) = \log k + \max_{g, \hat{d}} \log \hat{c} + \hat{V}_{\hat{b}}(\hat{b}, ND, S)\dot{\hat{b}} + \hat{V}_S(\hat{b}, ND, S) + \frac{g}{\rho}$$

$$\text{s.t. } \hat{c} = r^k(S) - \delta + \hat{d} - g$$

$$\dot{\hat{b}} = \rho \hat{b} + \hat{d} - \hat{b}g$$

$$\hat{b} \leq m(\tau)$$

which verifies (21).

The detrending property (21) provides a simple way to express the value function at states where the household's debt has a jump from some (detrended) vale \hat{b} to some \hat{b}_+ . To see this, take any k, b such that $\hat{b} = \frac{b}{k}$. To make \hat{b} jump to \hat{b}_+ the household needs to choose Δ so that

$$\Delta = \frac{\hat{b}_+ - \hat{b}}{1 - \hat{b}_+} \cdot k$$

which implies that the capital stock jumps from k to $k_+ = k + \Delta$, and the un-detrended debt from b to $k_+\hat{b}_+$, with

$$k_+ = \frac{1 - \hat{b}}{1 - \hat{b}_+} k \tag{33}$$

Therefore,

$$V^{ND}(k, b, ND, S) = V^{ND}(k_+, k_+ \hat{b}_+, ND, S)$$

A direct application of (21) finally gives

$$\hat{V}^{ND}(\hat{b}, S) = \hat{V}^{ND}(\hat{b}_+, S) + \log \frac{1 - \hat{b}}{1 - \hat{b}_+} \quad (34)$$

Deriving the model with jumps from a continuous model. Let us add a constraint $\hat{b} \leq \bar{D}$ to the maximization problem defining \hat{V}^{ND} , for some $0 < \bar{D} < \infty$. In this way, jumps in \hat{b} are not possible, because the control variable $\dot{\hat{d}}$ has to be finite. Now, if at some (normalized) time zero the household wants to move as quickly as possible her leverage from a value \hat{b} to a value $\hat{b}_+ > \hat{b}$, she has no choice but to choose $\dot{\hat{b}} = \bar{D}$ for a length of time $\epsilon > 0$ defined implicitly by

$$\hat{b}_+ = \hat{b} + \bar{D}\epsilon$$

For any $0 \leq \tau \leq \epsilon$ the first order condition with respect to $g(\tau)$ gives the usual optimal growth rate for the individual capital

$$g(\tau) = \frac{r^k(\tau) - \delta - \rho + \bar{D}}{1 - \hat{b}(\tau)} = \frac{r^k(\tau) - \delta - \rho + \bar{D}}{1 - \hat{b} - \tau\bar{D}}$$

Recall that $r^k(\tau)$ decreases over time, hence $r^k(0) \leq r^k(\tau) \leq r^k(\epsilon)$ for any $\tau \in [0, \epsilon]$. Since $\epsilon = \frac{\hat{b}_+ - \hat{b}}{\bar{D}}$ we obtain

$$\frac{k_+}{k} \leq \exp \left\{ \int_0^\epsilon \frac{r^k(0) - \delta - \rho + \bar{D}}{1 - \hat{b} - \tau\bar{D}} d\tau \right\} = \exp \left\{ \left[1 + \frac{r^k(0) - \delta - \rho}{\bar{D}} \right] \log \frac{1 - \hat{b}}{1 - \hat{b}_+} \right\}$$

and

$$\frac{k_+}{k} \geq \exp \left\{ \int_0^\epsilon \frac{r^k(\epsilon) - \delta - \rho + \bar{D}}{1 - \hat{b} - \tau\bar{D}} d\tau \right\} = \exp \left\{ \left[1 + \frac{r^k(\epsilon) - \delta - \rho}{\bar{D}} \right] \log \frac{1 - \hat{b}}{1 - \hat{b}_+} \right\}$$

Taking the limit for $\bar{D} \rightarrow \infty$ (or equivalently $\epsilon \rightarrow 0$) we conclude that

$$\log \frac{k_+}{k} \rightarrow \frac{1 - \hat{b}}{1 - \hat{b}_+}$$

which is in fact the formulation of a jump provided in (33).

Appendix B

Proof of Proposition 1.

As in Eaton and Gersovitz (1981) it is easy to show that the value of not defaulting $\hat{V}^{ND}(\hat{b}, ND, S)$ is continuous and strictly decreasing \hat{b} , while the value of defaulting $\hat{V}^D(\hat{b}, ND, S)$ is independent on \hat{b} . If $m^*(S)$ represents endogenous borrowing constraints then $\hat{V}^{ND}(m^*(S), ND, S) \leq \hat{V}^D(m^*(S), ND, S)$, because otherwise there would exist a leverage $\hat{b} = m^*(S) + \epsilon$, with $\epsilon > 0$, such that $\hat{V}^{ND}(\hat{b} + \epsilon, ND, S) > \hat{V}^D(\hat{b} + \epsilon, ND, S)$, a contradiction to the definition of endogenous borrowing constraints. Since borrowing constraint are sustainable only if $\hat{V}^{ND} \geq \hat{V}^D$, we conclude that if $m^*(S)$ is endogenous then (25) holds. Now assume that (25) holds, we want to prove that $m^*(S)$ is endogenous. Again, by strict monotonicity of \hat{V}^{ND} in \hat{b} and independence of \hat{V}^D on \hat{b} , we have $\hat{V}^{ND}(\hat{b}, ND, S) < \hat{V}^D(1, \hat{b}, ND, S)$ for all $\hat{b} > \hat{m}^*(S)$, so that $m^*(S)$ is endogenous.

Appendix C

This section derives the differential equations governing the evolution of the marginal product of capital r^k , of the aggregate capital K and of the fraction ϕ of aggregate capital allocated to households in the default state.

Evolution of r^k . From (2) we have

$$\dot{w} = (1 + \gamma)(w - \underline{w} + \gamma) \frac{\dot{H}}{H} \quad (35)$$

Aggregating k and h across households in (20) gives

$$\frac{\dot{H}}{H} = \frac{\dot{K}}{K} - \frac{1}{\alpha} \frac{\dot{w}}{w} \quad (36)$$

and from (4)

$$\frac{\dot{r}^k}{r^k} = -\frac{1 - \alpha}{\alpha} \frac{\dot{w}}{w} \quad (37)$$

Substituting (35) into (36) and then into (37) gives (23), where

$$a(r^k) = \frac{(1 - \alpha)(1 + \gamma)(w - \underline{w} + \gamma)}{\alpha w + (1 + \gamma)(w - \underline{w} + \gamma)}$$

Evolution of K . At discretized intervals of length ϵ we have,

$$K_{t+\epsilon} = (1 - \phi_t) K_t e^{\int_0^\epsilon g_{t+\tau} d\tau} + \phi_t K_t \left[e^{-\theta\epsilon} e^{\int_0^\epsilon g_{t+\tau}^d d\tau} + \int_0^\epsilon \frac{e^{\int_0^\tau g_{t+T}^d dT + \int_\tau^\epsilon g_{t+T} dT}}{1 - m_{t+\tau}} \theta e^{-\theta\tau} d\tau \right]$$

which gives

$$K_{t+\epsilon} = K_t \left[(1 - \phi_t)(1 + g_t\epsilon) + \phi_t \left[1 + (g_t^d - \theta)\epsilon + \frac{\theta}{1 - m_t}\epsilon \right] \right] + o(\epsilon)$$

Taking the limit for $\epsilon \rightarrow 0$ we obtain (39).

Evolution of ϕ At discretized intervals of length ϵ we have

$$\begin{aligned} 1 - \phi_{t+\epsilon} &= \frac{K_{t+\epsilon} - \phi_t K_t e^{-\theta\epsilon} e^{\int_0^\epsilon g_{t+\tau}^d d\tau}}{K_{t+\epsilon}} \\ \phi_{t+\epsilon} &= \phi_t \frac{e^{-\theta\epsilon + \int_0^\epsilon g_{t+\tau}^d d\tau}}{e^{\int_0^\epsilon G_{t+\tau} d\tau}} \\ \phi_{t+\epsilon} &= \phi_t [1 + (g_t^d - \theta - G_t)\epsilon] + o(\epsilon) \end{aligned}$$

from which we get (40).

Appendix D

Proof of Proposition 3.

I will prove that, for any realization of \tilde{t} , and for $z \in \{BB, CC\}$ the optimal consumption paths $\{c_t^{*z}\}$, respectively in the boom-bust and in the capital control equilibrium, satisfy $c_t^{*BB} > c_t^{*CC}$ for any $t > 0$. Take any initial $\hat{b}_0 \leq m^{CC}(\underline{w}) = m_+^{*CC}(\underline{w})$. In the boom-bust equilibrium the household optimally chooses a leverage jump to $m^{*BB}(\underline{w})$ so that capital jumps from k_0 to $k_+ = \frac{1-\hat{b}_0}{1-m^{*BB}(\underline{w})}$. Consumption at time zero is $c_0^* = \hat{c}_0^{*BB}k_+ = \rho[1 - m^{*BB}(\underline{w})]k_+ = \rho(1 - \hat{b}_0)$. From time zero to the turning point consumption grows at the rate $G^{*BB} = \frac{r^k(\underline{w})-\delta-\rho}{1-(1-\eta)m^{*BB}(\underline{w})}$. Similarly, in the CC equilibrium the leverage jumps to m^{CC} , consumption at time zero is again c_0^* , and consumption grows at rate $G^{*CC} = \frac{r^k(\underline{w})-\delta-\rho}{1-m^{CC}(\underline{w})}$. Since $\hat{V}_{CC}^{ND}(m^{CC}, ND, \underline{w}) > \hat{V}_{CC}^D(m^{CC}, ND, \underline{w})$, we can find m such that $m_+^{*CC} < m < m^{*BB}$ and satisfying $\log \hat{c}^{*CC} + \frac{G^{*CC}}{\rho} = \log \log c^{*D} + \frac{g^{*D}}{\rho} + \frac{\theta}{\rho} \log \frac{1}{1-m}$. We know that $\log \hat{c}^{*BB} + \frac{G^{*BB}}{\rho} = \log \hat{c}^{*D} + \frac{g^{*D}}{\rho} + \frac{\theta}{\rho} \log \frac{1}{1-m^{*BB}}$. Since $\hat{c}^{*BB} < \hat{c}^{*CC}$, it must be the case that $G^{*BB} > G^{*CC}$. The time \tilde{t}^z of the turning point for the equilibrium $z = \{BB, CC\}$ is $\tilde{t}^z = \frac{\log \bar{H} - \log H_0}{G^{*z}}$ and the household's capital stocks at the turning points are $k_{\tilde{t}^{BB}} = e^{\log \bar{H} - \log H_0} \frac{1-\hat{b}_0}{1-m^{*BB}(\underline{w})}$ and $k_{\tilde{t}^{CC}} = e^{\log \bar{H} - \log H_0} \frac{1-\hat{b}_0}{1-m^{CC}(\underline{w})}$. Notice that $k_{\tilde{t}^{BB}} > k_{\tilde{t}^{CC}}$. Without loss of generality assume that the household defaults at the turning point also in the CC equilibrium (the no-default choice would give the same utility). For the two equilibria z , let's compare the value of the consumption streams $\{c_{T^z(\tau)}^{*D}\}_{\tau=0}^{\infty}$ at any $T^z(\tau) = \tilde{t}^z + \tau$, where $\tau > 0$ is the time elapsed since \tilde{t}^z . For any $\tau > 0$ the marginal return to capital at time $T^z(\tau)$ in the two equilibria satisfies $r_{T^{BB}}^{k, BB} > r_{T^{CC}}^{k, CC}$. This is because at any state the growth rate of capital $G^{*z}(S)$ is strictly bigger in the CC state, $G_K^{*BB}(S) = [1 - \xi(S)]r^k(S) - \delta - \rho < G_K^{*CC} = \frac{r^k(S)-\delta-\rho}{1-m^{*CC}(S)}$. Therefore for any $\tau > 0$, the growth rate of the household's capital in equilibrium z is such that $g_{T^{BB}(\tau)}^{*D} > g_{T^{CC}(\tau)}^{*D}$. Given that $k_{\tilde{t}^{BB}} > k_{\tilde{t}^{CC}}$ we conclude that $k_{T^{BB}(\tau)} > k_{T^{CC}(\tau)}$ for any $\tau \geq 0$. By (31) we then have $c_{T^{BB}(\tau)}^{*D} > c_{T^{CC}(\tau)}^{*D}$ for any $\tau \geq 0$. In conclusion, consumption in the BB equilibrium is strictly higher than that in the CC equilibrium at any time, implying that the BB equilibrium provides strictly higher utility.

Appendix E

Proof of Proposition 2.

Step 1: At states where $m^(S)$ binds, derive a system of differential equations characterizing $m^*(S)$. At such states, equations (25)-(34) together imply¹⁸(27). Taking the first order condition for \hat{V}^j with respect to consumption and capital growth, we obtain equations*

¹⁸Equation (25) and the optimality of a binding borrowing constrain imply that $\frac{d\hat{V}^D(m^*, S)}{dt} = \frac{d\hat{V}^{ND}(m^*, S)}{dt}$ or, equivalently, $\hat{V}_S^D \dot{S} = \hat{V}_b^{ND} \dot{m} + \hat{V}_S^{ND} \dot{S}$.

(28)-(31). The right-hand side of (27) is also obtained using (34), and incorporates the value $\log \frac{1}{1-m^*}$ of the (detrended) jump (see (33)) in the capital stock when the household re-enters the financial market with $\hat{b} = 0$ and immediately follows the optimal policy $\hat{b}_+^* = m^*$.

Recall that after the turning point $\pi^*(S) = 0$. Substituting (28)-(31) into (27) we obtain the following differential equation

$$\dot{m} = -(1 - m^*)(\rho + \theta) \log(1 - m^*) - m^*[(1 - \xi)r^k - \delta - \rho] - \xi r^k \quad (38)$$

The evolution of the marginal product of capital r^k after the turning point is given by (23), where the growth rate of aggregate capital G_K^* is now a weighted average of the growth rate of the capital of households in the default and in the non default states. In particular,

$$G_K^* = (1 - \phi^*)g^* + \phi^* \left(g^{*D} + \theta \frac{m^*}{1 - m^*} \right) \quad (39)$$

with the weight ϕ^* representing the fraction of aggregate capital owned by households in the default state. The evolution of ϕ^* is given by¹⁹,

$$\dot{\phi}^* = -\phi^*(\theta + G_K^* - g^{*D}) \quad (40)$$

Equations (23), (38), (39) define a system of three first order differential equations in the variables m , r^k , ϕ for the equilibrium after the turning point. The initial condition for r^k must be $r(\underline{w})$ for any equilibrium, while for ϕ it is $\phi = 1$ in a boom-bust equilibrium and $\phi = 0$ in a smooth equilibrium.

Step 2. Define an upper bound m_t^L for m_t^ at any $t > \tilde{t}$.* From (38) we can write $\dot{m} = f(m; r^k)$. Condition (7) guarantees that f has an inverted-U shape with two solutions $m^L(r^k) < m^H(r^k)$, as shown in Figure 6. By the assumption $\xi'(w) \leq 0$, the cost ξ increases in r^k and thus it is easy to show that f decreases in r^k . Therefore, as $r^k \downarrow \rho + \delta$ monotonically after the turning point we have $m^H(r^k) \uparrow 1$ monotonically and $m^L(r^k)$ decreases monotonically. It follows that any solution to the system of differential equations must have an initial leverage $m_+^* \leq m^L(r^k(\underline{w}))$, because any other leverage $m_+ > m^L(r^k(\underline{w}))$ would lead to an increasing path $m \rightarrow 1$ (this cannot be a solution since $m = 1$ would imply $\hat{c}^* = 0$). Notice that $m^L(r^k(\underline{w}))$, the upper bound for m_+^* , is not a function of γ .

As we show in Step 6, in a boom-bust equilibrium the borrowing constraint $m^*(\underline{w})$ is binding before the turning point. The value of $m^*(\underline{w})$ can be computed as follows. Substitute the conditions $r^* = \rho + \eta g^*$ and $\dot{m} = 0$ into (28), and express g^* in terms of m^* and of the exogenous parameters ρ , δ , $r^k(\underline{w})$. Substitute g^* and equations (29)-(31) into (27) we obtain an equation in m^* , where γ does not appear. For η low enough, the largest solution m^* for this equation satisfies $m^* > m^L(r^k(\underline{w}))$, and this solution represents the endogenous borrowing constraint $m^*(\underline{w})$, which is then also independent of γ . In conclusion, the value of γ does not affect the upper bound $m^L(r^k(\underline{w}))$ for m_+^* , nor it affects $m^*(\underline{w})$. We conclude that $\gamma > 0$ does not determine the existence of a boom-bust equilibrium.

Step 3. Prove the existence of a solution to the system of differential equations. Define

¹⁹Appendix C shows how (39) and (40) are obtained.

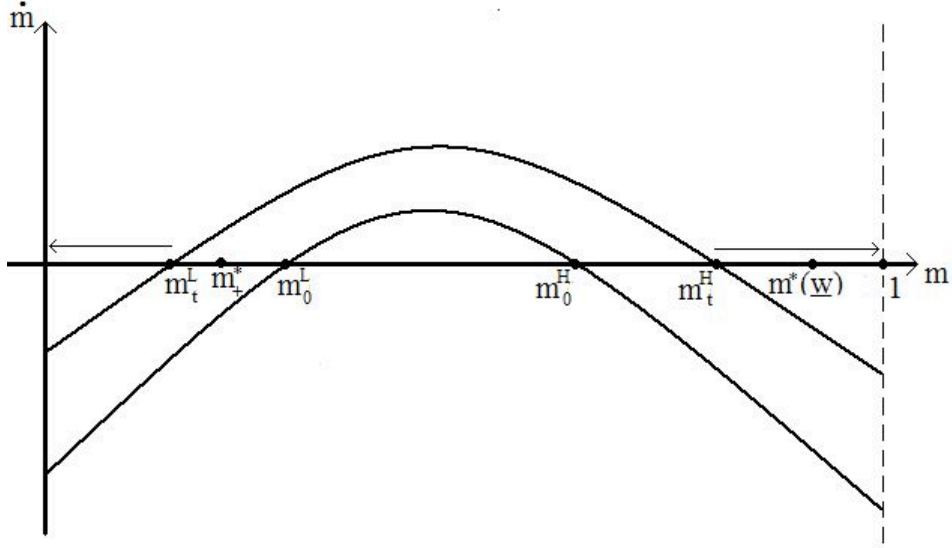


Figure 6: Phase diagram for the evolution of m_t . Arrows show how m_t^L and m_t^H evolve over time. A binding constraint $m^*(\underline{w})$ implies default at the turning point, when the leverage constraint jumps down to some $m_+^*(\underline{w}) < m_0^L$. Since $m^*(\underline{w})$ and m_0^L don't depend on γ , the existence of a boom-bust equilibrium, and the minimum size $m^*(\underline{w}) - m_0^L$ for the sudden stop, are independent on $\gamma > 0$.

$r_0 = r^k(\underline{w})$ and choose any $\phi_0 \in [0, 1]$. I show that the system (23), (38), (40) has a solution for the given initial conditions r_0, ϕ_0 . Write the system as $\dot{y} = z(y)$ with $y = (m, r^k, \phi)$ and let y_n and $z_n(y)$ be the n -th element in the vector y and $z(y)$ respectively. Define

$$Y = \{(m, r^k, \phi) | 0 \leq m \leq m^L(r_0), 0 \leq r^k \leq r_0, 0 \leq \phi \leq 1\}$$

For $p > 0$ and small, denote with $B(y; p)$ the open ball centered at y with radius p and define \bar{Y}_0 as

$$\bar{Y}_0 = \bigcup_{y \in Y} B(y, p)$$

It is easy to show that there exists a constant $L > 0$

$$|z_n(y)| \leq L \quad \forall y \in \bar{Y}_0$$

with $n = 1, 2, 3$. Moreover, call $J_{i,j}(y)$ the typical element of the the Jacobian matrix of $z(y)$ at any differentiable point y . We can show that there is a constant $L_1 > 0$ such that

$$|J_{i,j}(y)| \leq L_1 \quad \forall y \in \bar{Y}_0$$

$i = 1, 2, 3, j = 1, 2, 3$. Then the following Lipschitz condition holds

$$\sum_n |z_n(y) - z_n(x)| \leq L_1 \sum_n |y_n - x_n|$$

for any pair $y, x \in \bar{Y}_0$. Define the constant $T_1 = \frac{p}{L}$. General arguments on the existence of local solutions²⁰ to systems of differential equations imply that, for any $y \in \bar{Y}_0$ there exists one and only one continuous solution $y^*(\tau)$ with

$$y^*(0) = y$$

$$|y_n^*(\tau) - y_n^*(0)| \leq p \quad 0 \leq \tau \leq T_1$$

$n = 1, 2, 3$. Let $F_n(y^*(0)) = y_n^*(T_1)$ be the continuous function that maps the initial condition $y^*(0)$ to the n -th component of the final value $y^*(T_1)$. Define the set $M_0 = [0, m^L(r_0)]$ and choose initial states $\underline{y}^{0*}(0) = (0, r_0, \phi_0)$ and $\bar{y}^{0*}(0) = (m^L(r_0), r_0, \phi_0)$. Notice that $\underline{y}^{0*}(0), \bar{y}^{0*}(0) \in Y$ and that $z_1(\underline{y}^{0*}(0)) < 0, z_1(\bar{y}^{0*}(0)) \geq 0$. Our discussion in Step 1 about the shape of f then implies $F_1(\underline{y}^{0*}(0)) < 0$ and $F_1^1(\bar{y}^{0*}(0)) > m^L(r_0)$. By continuity of F there exists a closed interval $\bar{M}_1 = [\underline{m}_1, \bar{m}_1] \subset M_0$ such that $F_1(m, r_0, \phi_0) \in M_0 \forall m \in \bar{M}_1$, and $F_1(\bar{y}^{1*}(0)) = m^L(r_0)$ and $F_1(\underline{y}^{1*}(0)) = 0$ for initial conditions $\bar{y}^{1*}(0) = (\bar{m}_1, r_0, \phi_0)$, $\underline{y}^{1*}(0) = (\underline{m}_1, r_0, \phi_0)$. At any $\tau \in [0, T_1]$ we have $y_2^*(\tau) \in [\rho + \delta, r_0]$ (see assumption (9)) and $y_3^*(\tau) \in [0, 1]$ for any $y_1^*(0) \in \bar{M}_1$.

The procedure in Step 2 can be iterated as follows. Define the set \bar{Y}_{T_1} as

$$\bar{Y}_{T_1} = \{y | y = F(y^*(0)), y^*(0) = (m, r_0, \phi_0), m \in \bar{M}_1\}$$

Define $T_2 = T_1 + \frac{p}{L}$. Since $\bar{Y}_{T_1} \subset \bar{Y}_0$, we then know that, for any $y \in \bar{Y}_{T_1}$ there exists a unique solution $y^*(\tau)$ with $\tau \in [T_1, T_2]$ to system of differential equations such that $y^*(T_1) = y$. In this way we have constructed a solution $y^*(\tau)$ for $\tau \in [0, T_2]$ and for any initial $y^*(0) = (m, r_0, \phi)$ such that $m \in \bar{M}_1$. Clearly, for any such solution we have $y^*(T_2) = F(F(y^*(0)))$. Again, we can construct an interval $\bar{M}_2 = [\underline{m}_2, \bar{m}_2] \subset \bar{M}_1$ such that $F_1(F(m, r_0, \phi_0)) \in M_0 \forall m \in \bar{M}_2$, and $F_1(F(\bar{y}^{2*}(0))) = m^L(r_0)$ and $F_1(F(\underline{y}^{2*}(0))) = 0$ for initial conditions $\bar{y}^{2*}(0) = (\bar{m}_2, r_0, \phi_0)$, $\underline{y}^{2*}(0) = (\underline{m}_2, r_0, \phi_0)$. Proceeding in this way we construct a sequence $M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ of nested, non-empty closed sets. Choose any $N = 1, 2, \dots$, then any $m \in M_{N+1}$ gives a valid initial condition $y^*(0) = (m, r_0, \phi_0)$ to which we associate a unique solution $y^*(\tau)$ to the system up to time $\tau = T_N$ such that $y_1^*(T_N) \in [0, m^L(r_0)]$. By the finite intersection property we conclude that

$$M_+ \equiv \bigcap_{N=1}^{\infty} M_N \neq \emptyset$$

Any $m_+^* \in M_+$ gives an initial condition $y^*(0) = (m_+^*, r_0, \phi_0)$ and a corresponding solution $y^*(\tau)$ for $\tau \geq 0$ to the system of differential equations.

Step 4. Show that $m_+(\underline{w}) < m^L(r_0)$. By Step 1 we already know that $m_+^* \leq m^L(r_0)$. Suppose, by contradiction, that $m_+^* = m^L(r_0)$. By the definition of $m^L(r_0)$, we would then have $\dot{m}^*(0) = 0$. Using (38) we would obtain

$$\ddot{m}^*(0) = -\dot{r}^k(0)[m_+^* + \xi(0)(1 - m_+^*)] - \dot{\xi}(0)r^k(0)(1 - m_+^*) > 0$$

²⁰See, for instance, Franklin (1954).

Therefore, $\dot{m}^*(\tau) > 0$ in a neighborhood of $\tau = 0$, which contradicts $m^*(\tau) \leq m^L(r_0)$ for all τ .

Step 5. Derive the endogenous borrowing constraint $m^(\underline{w})$ before the turning point and show that $m^*(\underline{w}) > m_+^*(\underline{w})$.* The balanced growth before the turning point is given by

$$G_{K,\phi} = (1 - \phi) \frac{r_0 - \delta - \rho}{1 - m_+^\phi} + \phi \frac{r_0 - \delta - \rho - \pi_\phi m^*}{1 - m^*}$$

The balanced growth $G_{k,\phi}$ is a weighted average of capital growth of households that choose the endogenous leverage $m_+^\phi = m_+^*(\underline{w})$ defined in the previous step (where we make explicit the dependence on ϕ), and of capital growth for households that choose a leverage $m^* > m_+^\phi$ and that therefore default at the turning point. Our goal is to show that m^* corresponds to the endogenous borrowing constraint before the turning point. As usual, the arrival rate of the turning point is $\pi_\phi = \eta G_{K,\phi}$.

As mentioned in section 5, the leverage levels that we need to consider as candidates for the optimal household's policy are either m^* or m_+^ϕ . Recall also that m^* must be characterized by the property of making the household indifferent between defaulting and not defaulting. Therefore, to construct the equilibrium values m^* , m_+^ϕ and ϕ it is enough to compare the values of the following five strategies in the first stage: *a)* given a leverage m^* , have the leverage first jump down to, and then remain constant at m_+^ϕ , *b)* keep the constant leverage m^* *c)* default and, after readmission to the financial market, keep a leverage m_+^ϕ , *d)* default, jump to m^* after regaining admission to the financial market, instantaneously default and keep repeating the cycle *e)* keep a constant leverage m_+^ϕ . Call the values corresponding to the strategies *a)*-*e)*, respectively, $V_{m_+^\phi, \phi}^{ND}(m^*)$, $V_\phi^{ND}(m^*)$, $V_{m_+^\phi, \phi}^D$, $V_{m^*, \phi}^D$, $V_\phi^{ND}(m_+^\phi)$. Since the borrowing constraint is endogenous after the turning point, all the strategies above yield the same value $V_{TP,\phi}$ at the turning point (the household's value at the turning point is independent on her default choice). We have,

$$V_{m_+^\phi, \phi}^{ND}(m^*) = V_\phi^{ND}(m_+^\phi) - \frac{1}{\rho} \log \frac{1 - m_+^\phi}{1 - m^*} \quad (41)$$

$$(\rho + \pi_\phi + \theta) V_{m^*, \phi}^D = \log \hat{c}^D + \frac{g^D}{\rho} + \theta V_{m^*, \phi}^D + \frac{\theta}{\rho} \log \frac{1}{1 - m^*} + \pi_\phi V_{TP,\phi} \quad (42)$$

$$(\rho + \pi_\phi + \theta) V_{m_+^\phi, \phi}^D = \log \hat{c}^D + \frac{g^D}{\rho} + \theta V_{m_+^\phi, \phi}^D + \frac{\theta}{\rho} \log \frac{1}{1 - m_+^\phi} + \pi_\phi V_{TP,\phi} \quad (43)$$

$$(\rho + \pi_\phi) V_\phi^{ND}(\hat{b}_1) = \log \hat{c} + \frac{g}{\rho} + \pi_\phi V_{TP,\phi} \quad (44)$$

with \hat{c} , g , \hat{c}^D , g^D as in (30)-(29). Define $\hat{b}_\phi > m_+^\phi$ the unique value solving

$$V_{m_+^\phi, \phi}^{ND}(\hat{b}_\phi) = V_{m_+^\phi, \phi}^D \quad (45)$$

We will start guessing that $m^* = \hat{b}_\phi$. In other words, we start by guessing that the household is indifferent between the option of keeping a leverage equal to the endogenous constraint

m^* , and the option of repaying part of her debt in order to make the leverage jump down from m^* to m_+^ϕ , where no default premium is paid. In this case, equations (45)-(43) imply that,

$$V_{m_+^\phi, \phi}^{ND}(m^*) = V_{m_+^\phi, \phi}^D = V_{m^*, \phi}^D$$

Notice that any household with leverage $\hat{b} > m^*$ is always better-off by defaulting, as required by the definition of endogenous borrowing constraint. Consider the following condition,

$$V_{m_+^\phi, \phi}^{ND}(m^*) \geq V_\phi^{ND}(m^*) \quad (46)$$

If (46) holds for $\phi = 0$ then we have found the endogenous borrowing constraint before the turning point in a smooth equilibrium. In fact, households strictly prefer to default if and only if they have an initial leverage $\hat{b} > m^*$, but otherwise prefer to adjust their leverage down to m_+^ϕ . If (46) does not hold for $\phi = 0$, it might hold with equality for some $\phi \in (0, 1]$. In this case we have found the endogenous borrowing constraint in an equilibrium where, before the turning point, a constant fraction ϕ of capital is allocated to households with leverage m^* . Notice that households can indifferently jump between the leverages m_+^* and m^* .

The final possibility (notice that value functions in (46) are continuous in ϕ) is that (46) does not hold for any $\phi \in [0, 1]$. This means that households always strictly prefer to lever up to the endogenous limit m^* , and default altogether at the turning point (boom-bust equilibrium). In this case we make a new guess for m^* . Fix $\phi = 1$ and notice that for $m^* \geq \hat{b}_1$,

$$\rho [V_1^{ND}(m^*) - V_{m^*, 1}^D] = (\rho + \theta) \log(1 - m^*) + \frac{m^*}{1 - m^*} [r_0 - \delta - \rho - \pi_1 m] + \xi(\underline{w}) r_0$$

where we have $\pi_1 = \eta \frac{r_0 - \delta - \rho}{1 - (1 - \eta)m^*}$. It is easy to show that $V_1^{ND}(m^*) - V_{m^*, 1}^D < 0$ for m^* sufficiently close to one, while, since (46) does not hold, $V_1^{ND}(m^*) - V_{m^*, 1}^D > 0$ for $m^* = \hat{b}_1$. Hence, by continuity, there exists a $m^* > \hat{b}_1$ such that $V_1^{ND}(m^*) - V_{m^*, 1}^D = 0$, i.e. (25) holds. We have found an endogenous borrowing constraint before the turning point of a boom-bust equilibrium.

By Step 4 we have $m_+^* < m^L(r_0)$ and thus $V_{m_+^\phi}^{ND}(m_+^\phi) > V_{m_+^\phi, \phi}^D$, implying that $\hat{b}_\phi > m_+^\phi$ for all ϕ . Since the endogenous borrowing constraint that we have constructed must satisfy $m^* \geq \hat{b}_\phi$, it follows that $m^* > m_+^*$, implying that there is always a credit crunch at the turning point.

Step 6. Show that bounds $\underline{\eta}$, $\bar{\eta}$ exist, and that $\underline{\eta}$ does not depend on γ . Set $\phi = 1$ and notice that

$$V_1^{ND}(m) - V_{m, 1}^D \rightarrow f(m, r_0) \quad \text{for } \eta \rightarrow 0$$

where convergence is pointwise and f was defined in Step 1. Recall that $f(m^H(r_0), r_0) = 0$. Then, by continuity, for any η sufficiently small there is an endogenous borrowing constraint m^* satisfying $V_1^{ND}(m^*) = V_{m^*, 1}^D$ and arbitrarily close to $m^H(r_0)$. Since $m^H(r_0) > m_+^1$ we have constructed $m^*(\underline{w})$ in a boom bust equilibrium. Call $\underline{\eta}$ the minimum value of η for

which such boom bust equilibrium exists. We can express $V_1^{ND}(m^*) - V_{m^*,1}^D$ using first order conditions to substitute for c^*, c^{*D}, g^*, g^{*D} and writing π^* in terms of m^* . The value for m^* can then be computed analytically from $V_1^{ND}(m^*) - V_{m^*,1}^D = 0$. Since $V_1^{ND}(m^*) - V_{m^*,1}^D$ is independent on $V_{TP,1}$, then also the solution m^* is independent on $V_{TP,1}$ and thus on γ ²¹.

Finally, set $\phi = 0$ and take η arbitrarily large. No household would choose an equilibrium leverage strictly greater than m_+^0 , because the default premium $\pi_0 = \eta G_{K,0}$ would be arbitrarily large. It follow that for any η large enough only a smooth equilibrium exists.

²¹Values of $m_+^*(\underline{w})$ and of $m^*(S)$ a states $S > \underline{w}$ depend on γ .